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INCOMPRESSIBLE LIMIT FOR SOLUTIONS OF THE ISENTROPIC NAVIER–STOKES EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS

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ABSTRACT. – We study here the limit of global weak solutions of the compressible Navier–Stokes equations (in the isentropic regime) in a bounded domain, with Dirichlet boundary conditions on the velocity, as the Mach number goes to 0. We show that the velocity converges weakly in L^2 to a global weak solution of the incompressible Navier–Stokes equations. Moreover, the convergence in L^2 is strong under some geometrical assumption on Ω. © Elsevier, Paris

1. Introduction

This paper is devoted to the study of the so-called incompressible limit for solutions of the compressible isentropic Navier–Stokes equations. We consider here the case of a flow in a bounded domain with the natural physical boundary conditions for a viscous fluid, namely (homogeneous) Dirichlet boundary conditions. This work is, in some sense, the sequel of [15] where various cases were considered, namely the cases of a periodic flow, of a flow in the whole space, or in a bounded domain with other boundary conditions than Dirichlet conditions. In the case of Dirichlet boundary conditions, the results of [15] only cover situations where the initial conditions are "well prepared" (density "almost" constant, velocity "almost" incompressible).

Before describing more precisely our main mathematical result, we wish to describe it and emphasize a new striking phenomenon caused by the boundary conditions. As is well known physically, one expects that, as the Mach number goes to 0, fast acoustic waves are generated carrying the energy of the potential part of the flow (and a normalized part of the internal energy of the gas). For periodic flows (or for some particular boundary conditions), these waves subsist forever and their frequency grows. Mathematically speaking, this means that the solutions of compressible (isentropic) Navier–Stokes equations may only converge *weakly* (in L^2) to the solutions of incompressible Navier–Stokes equations—and they do as shown in [15]. However, in the case of a viscous flow in a bounded domain with the usual Dirichlet boundary condition, under a generic assumption, we shall show here that the acoustic waves are instantaneously (asymptotically) damped, due to the formation of a thin boundary layer. This layer dissipates the energy carried by the waves and, from a mathematical viewpoint, we obtain a *strong* convergence in L^2. This phenomenon is also present for a related problem which presents, as is classical, striking analogies with the one studied here, namely the so-called "Ekman pumping" for rotating fluids (see for instance [7]).

Let us now say a word on the mathematical method used in [15] and here. In [15], the main tool was the use of the group generated by the wave operator (corresponding to the acoustic waves alluded to above), a method introduced in earlier works ([19,6], and [17]) which requires some smoothness of solutions. It turns out (see [15] for more details) that there is just enough regularity information to go globally in time from weak solutions of compressible equations to the ones of incompressible equations. Here however, this method does not apply because of subtle interactions between dissipative effects and wave propagation near the boundary. These two phenomena can not be split as in [15] and the group generated by the wave operator (with slip boundary conditions) does not seem to be adapted to this problem. The proof therefore relies upon spectral analysis of the semigroup generated by the *dissipative* wave operator, together with Duhamel's principle. Let us also mention some earlier related works ([9,20,10,8]) which concern strong solutions (on a finite time interval) in the whole space, and the forthcoming paper [4] where the strong convergence is shown in the whole space using the dispersion of acoustic waves.

We may now state precisely our main result for which we need to introduce some notations. We thus consider a viscous isentropic compressible fluid with a γ pressure law ($p = a\rho^\gamma$, $\gamma > 1$) as in [15]. The incompressible limit arises in an appropriate scaling such that we may write equations for the density $\rho^\varepsilon \geqslant 0$ and the velocity $\mathbf{u}^\varepsilon$ as follows

$$\partial_t \rho^\varepsilon + \operatorname{div}\big(\rho^\varepsilon \mathbf{u}^\varepsilon\big) = 0 \quad \text{in } \Omega \times (0,\infty), \qquad \mathbf{u}^\varepsilon = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1}$$

$$\begin{aligned} &\partial_t \big(\rho^\varepsilon \mathbf{u}^\varepsilon\big) + \operatorname{div}\big(\rho^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon\big) - \mu \Delta \mathbf{u}^\varepsilon - (\lambda+\mu)\nabla \operatorname{div} \mathbf{u}^\varepsilon \\ &\qquad + \frac{\nabla(\rho^\varepsilon)^\gamma}{\gamma\varepsilon^2} = 0 \quad \text{in } \Omega \times (0,\infty), \end{aligned} \tag{2}$$

where Ω is a bounded smooth (say C^2) domain of $\mathbb{R}^d$ with $d = 2$ or 3. We impose some initial conditions

$$\rho^\varepsilon{}_{|t=0} = \rho_0^\varepsilon \geqslant 0, \qquad \rho^\varepsilon \mathbf{u}^\varepsilon{}_{|t=0} = \mathbf{m}_0^\varepsilon \quad \text{in } \Omega, \tag{3}$$

where, by convention, $\mathbf{m}_0^\varepsilon = 0$ on $\{\mathbf{x} \in \Omega \mid \rho_0^\varepsilon(\mathbf{x}) = 0\}$ and we assume

$$E_0^\varepsilon = \int_\Omega \left\{ \pi_0^\varepsilon + \frac{|\mathbf{m}_0^\varepsilon|^2}{2\rho_0^\varepsilon} \right\} \mathrm{d}\mathbf{x} \leqslant C. \tag{4}$$

Here and below, C denotes various positive constants independent of ε and we denote by:

$$\pi^\varepsilon = \frac{(\rho^\varepsilon)^\gamma - 1 - \gamma(\rho^\varepsilon - 1)}{\varepsilon^2\gamma(\gamma-1)} \quad \text{and} \quad \pi_0^\varepsilon = \frac{(\rho_0^\varepsilon)^\gamma - 1 - \gamma(\rho_0^\varepsilon - 1)}{\varepsilon^2\gamma(\gamma-1)}.$$

Finally, we set $|\mathbf{m}_0^\varepsilon|^2/\rho_0^\varepsilon = 0$ on $\{\mathbf{x} \in \Omega \mid \rho_0^\varepsilon(\mathbf{x}) = 0\}$.

The parameter $\varepsilon > 0$ corresponds to the Mach number and we wish to let it go to 0_+. For the sake of simplicity, the viscosity coefficients λ and μ are taken independent of ε and the reference density is equal to 1. Finally, we assume that $\mu > 0$, $\lambda + 2\mu > 0$, and that $(\rho_0^\varepsilon)^{-1/2}\mathbf{m}_0^\varepsilon$ converges weakly in $L^2(\Omega)^d$ to some $\mathbf{u}_0$.

The existence of global weak solutions to the above system (for a fixed $\varepsilon > 0$) was obtained by P.L. Lions in [12,14] under the condition:

$$\gamma \geqslant \gamma_0, \quad \text{where} \quad \gamma_0 = \frac{3}{2} \quad \text{if} \quad d=2 \quad \text{and} \quad \gamma_0 = \frac{9}{5} \quad \text{if} \quad d=3. \tag{5}$$

Let us observe at this point that most of the arguments in [12] carries over to the case when $\gamma > d/2$ for all d and that this restriction also plays a crucial role in [15] and here. Indeed, in all what follows, we assume that $\gamma > d/2$ and we shall not recall this condition. By a global weak solution of (1), (2), (3), we mean the following: $\rho^\varepsilon \in L^\infty(0,\infty; L^\gamma(\Omega))$, $\mathbf{u}^\varepsilon \in L^2(0,\infty; H_0^1(\Omega))^d$ solve (1), (2) in the sense of distributions. In addition, $\rho^\varepsilon \in C([0,\infty); L^p(\Omega))$ if $1 \leqslant p < \gamma$, $\rho^\varepsilon|\mathbf{u}^\varepsilon|^2 \in L^\infty(0,\infty; L^1(\Omega))$, $\rho^\varepsilon\mathbf{u}^\varepsilon \in C([0,\infty); L^{2\gamma/(\gamma+1)}(\Omega) - w)$, and (3) holds. Finally, we have the following energy inequality for almost all $t \geqslant 0$,

$$\int_\Omega \left\{\frac{1}{2}\rho^\varepsilon|\mathbf{u}^\varepsilon|^2 + \pi^\varepsilon\right\}(t)\,\mathrm{d}\mathbf{x} + \int_0^t\int_\Omega \left\{\mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)|\mathbf{u}|^2\right\}\mathrm{d}\mathbf{x}\,\mathrm{d}s \leqslant E_0^\varepsilon. \tag{6}$$

Let us also mention that in [12] the following additional bound is shown: $\rho^\varepsilon \in L^q_{\mathrm{loc}}([0,\infty)\times\Omega)$ where $q = \gamma - 1 + 2\gamma/d$. Without loss of generality, we may assume, in view of (6), that $\mathbf{u}^\varepsilon$ converges weakly in $L^2(0,\infty; H_0^1(\Omega))^d$ to some $\overline{\mathbf{u}}$.

Because of (6), ρ^ε converges to 1 (see [15] for more details). And we may thus expect from (1) that $\operatorname{div}\mathbf{u}^\varepsilon$ converges to 0, hence $\operatorname{div}\overline{\mathbf{u}} = 0$, and that $\overline{\mathbf{u}}$ thus solves the incompressible Navier–Stokes equations

$$\begin{aligned} &\partial_t\overline{\mathbf{u}} + \operatorname{div}(\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}) - \mu\Delta\overline{\mathbf{u}} + \nabla\pi = 0,\\ &\operatorname{div}\overline{\mathbf{u}} = 0 \quad \text{in } \Omega\times(0,\infty), \quad \text{and} \quad \overline{\mathbf{u}} = 0 \quad \text{on } \partial\Omega\times(0,\infty). \end{aligned} \tag{7}$$

This is indeed correct as shown in the following result and $\overline{\mathbf{u}}$ is a "Leray" solution, called below a global weak solution of (7), i.e.

$$\overline{\mathbf{u}} \in L^\infty\big(0,\infty; L^2(\Omega)\big)^d \cap C\big([0,\infty); L^p(\Omega)\big)^d \cap L^2\big(0,\infty; H_0^1(\Omega)\big)^d$$

(for all $p < 2$) solves (7) in the sense of distributions for some (say distribution) π. We finally introduce the usual decomposition of $\mathbf{u}_0$ into its irrotational and its potential parts namely $\mathbf{u}_0 = \overline{\mathbf{u}}_0 + \tilde{\mathbf{u}}_0$, where $\operatorname{div}\overline{\mathbf{u}}_0 = 0$, $\operatorname{curl}\tilde{\mathbf{u}}_0 = 0$ in Ω, and $\overline{\mathbf{u}}_0\cdot\mathbf{n} = 0$ on $\partial\Omega$, where $\mathbf{n}$ denotes the outward normal to $\partial\Omega$.

In order to state precisely our main theorem, we need to introduce a geometrical condition on Ω. Let us consider the following overdetermined problem:

$$-\Delta\phi = \lambda\phi \quad \text{in } \Omega, \qquad \frac{\partial\phi}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega, \quad \text{and } \phi \text{ is constant on } \partial\Omega. \tag{8}$$

A solution to (8) is said to be trivial if $\lambda = 0$ and ϕ is a constant. We will say that Ω satisfies assumption (H) if all the solutions of (8) are trivial. Schiffer's conjecture says that every Ω satisfies (H) excepted the ball (see for instance [5]). In two dimensionnal space, it is proved that every bounded, simply connected open set $\Omega \subset \mathbb{R}^2$ whose boundary is Lipschitz but not real analytic satisfies (H), hence property (H) is generic in $\mathbb{R}^2$.

Our main result reads as follows:

THEOREM 1. – *Under the above conditions, ρ^ε converges to* 1 *in* $C([0,T];L^\gamma(\Omega))$ *and* $\mathbf{u}^\varepsilon$ *converges to* $\overline{\mathbf{u}}$ *weakly in* $L^2((0,T)\times\Omega)^d$ *for all* $T>0$ *and strongly if* Ω *satisfies* (H). *In addition,* $\overline{\mathbf{u}}$ *is a global weak solution of the incompressible Navier–Stokes equations* (7) *satisfying* $\overline{\mathbf{u}}_{|t=0}=\overline{\mathbf{u}}_0$ *in* Ω.

More precisely, we will split the eigenvectors $(\Psi_{k,0})_{k\in\mathbb{N}}$ of the Laplace equation with Neumann boundary condition (which represent the acoustic eigenmodes in Ω) into two classes : those which are not constant on $\partial\Omega$ will generate boundary layers and will be quickly damped, thus converging strongly to 0; those which are constant on $\partial\Omega$ (nontrivial solutions of (8)), for which no boundary layer forms, will remain oscillating forever, leading to only weak convergence. Indeed, if (H) is not satisfied, $\mathbf{u}^\varepsilon$ will in general only converge weakly and not strongly to $\overline{\mathbf{u}}$ (like in the periodic case $\Omega=\mathbb{T}^d$ for instance). However, if at initial time $t=0$, no modes of second type are present in the velocity, the convergence to the incompressible solution is strong in L^2.

Theorem 1 also applies to $\Omega=\mathbb{T}^{d-1}\times[0,1]$, which does not satisfy (H). Notice that according to Schiffer's conjecture the convergence is not strong for general initial data when Ω is the two or three dimensionnal ball, but is expected to be always strong in any other domain.

2. Preliminaries

First, for any $(\mathbf{f},\mathbf{g})\in L^2(\Omega;\mathbb{C}^l)^2$ $(l\in\mathbb{N}^*)$, we denote the scalar product $\langle\mathbf{f}\mid\mathbf{g}\rangle=\int_\Omega\mathbf{f}\cdot\overline{\mathbf{g}}\,\mathrm{d}\mathbf{x}$. Let us also recall the definitions of Leray's projectors P on the space of divergence-free vector fields and Q on the space of gradients defined by

$$P=I-Q,\quad Q=\nabla\Delta_N^{-1}\mathrm{div},$$

where Δ_N^{-1} denotes the inverse Laplace operator with Neumann boundary conditions

$$f=\Delta_N^{-1}g,\quad \text{if } \Delta f=g \text{ in } \Omega,\qquad \frac{\partial f}{\partial\mathbf{n}}=0\quad \text{on } \partial\Omega,\quad \text{and}\quad \int_\Omega f\,\mathrm{d}\mathbf{x}=0.$$

Let $(\lambda_{k,0}^2)_{k\geqslant1}$ $(\lambda_{k,0}>0)$, be the nondecreasing sequence of eigenvalues and $(\Psi_{k,0})_{k\geqslant1}$ the orthonormal basis of $L^2(\Omega)$ functions with zero mean value of eigenvectors of the Laplace operator $-\Delta_N$ with homogenenous Neumann boundary conditions:

$$-\Delta\Psi_{k,0}=\lambda_{k,0}^2\Psi_{k,0}\quad \text{in } \Omega,\qquad \frac{\partial\Psi_{k,0}}{\partial\mathbf{n}}=0\quad \text{on } \partial\Omega. \tag{9}$$

Notice that it is possible to assume up to a slight modification of $(\Psi_{k,0})_{k\geqslant1}$ that if $\lambda_{k,0}=\lambda_{l,0}$ and $k\neq l$ then

$$\int_{\partial\Omega}\nabla\Psi_{k,0}\cdot\nabla\Psi_{l,0}\,\mathrm{d}\sigma=0. \tag{10}$$

Next, we recall a priori bounds on weak solutions derived in [15]. Let $\gamma>d/2$ and initial data $(\rho_0^\varepsilon,\mathbf{m}_0^\varepsilon)$ satisfying (3). Assuming that weak solutions $(\rho^\varepsilon,\mathbf{u}^\varepsilon)$ exist for all time (which is the case as soon as $\gamma\geqslant\gamma_0$), we deduce in view of the energy inequality (6) that π^ε and $\rho^\varepsilon|\mathbf{u}^\varepsilon|^2$ are bounded in $L^\infty(0,\infty;L^1(\Omega))$ and that $\mathbf{u}^\varepsilon$ is bounded in $L^2(0,\infty;H_0^1(\Omega))^d$ uniformly in ε.

Denoting $\kappa = \min(2, \gamma)$, we deduce as in [15] that

$$\sup_{t\geqslant 0}\left|\rho^\varepsilon - 1\right|_{L^\gamma(\Omega)} \leqslant C\varepsilon^{\kappa/\gamma} \quad \text{and} \quad \sup_{t\geqslant 0}\left|\rho^\varepsilon - 1\right|_{L^\kappa(\Omega)} \leqslant C\varepsilon. \tag{11}$$

Let us finally precise the time regularity of the velocity $\mathbf{u}^\varepsilon$. In Section 3.2, we will split $\mathbf{u}^\varepsilon = \mathbf{u}_1^\varepsilon + \mathbf{u}_2^\varepsilon$, where $\mathbf{u}_1^\varepsilon = \mathbf{u}^\varepsilon 1_{|\rho^\varepsilon - 1|\leqslant 1/2}$ and $\mathbf{u}_2^\varepsilon = \mathbf{u}^\varepsilon 1_{|\rho^\varepsilon - 1|> 1/2}$, which satisfy

$$\sup_{t\geqslant 0}\int_\Omega |\mathbf{u}_1^\varepsilon|^2\,\mathrm{d}\mathbf{x} \leqslant 2\sup_{t\geqslant 0}\int_\Omega \rho^\varepsilon|\mathbf{u}^\varepsilon|^2\,\mathrm{d}\mathbf{x} \leqslant 4C_0$$

and

$$\left|\mathbf{u}_2^\varepsilon\right|^2_{L^2(\Omega)} \leqslant 2\int_\Omega \left|\rho^\varepsilon - 1\right|\left|\mathbf{u}^\varepsilon\right|^2\mathrm{d}\mathbf{x} \leqslant C\varepsilon\left|\mathbf{u}^\varepsilon\right|^2_{L^{2\kappa/(\kappa-1)}(\Omega)} \leqslant C\varepsilon\left|\nabla\mathbf{u}^\varepsilon\right|^2_{L^2(\Omega)}.$$

Therefore, $\mathbf{u}_1^\varepsilon$ is bounded in $L^\infty(0,T;L^2(\Omega))^d$, whereas $\mathbf{u}_2^\varepsilon\varepsilon^{-1/2}$ is bounded in $L^2((0,T)\times\Omega))^d$ for all $T>0$.

3. Proof of Theorem 1

In view of the results of [15], we may assume up to the extraction of a subsequence that the incompressible part $P\mathbf{u}^\varepsilon$ converges strongly to some incompressible vector field $\bar{\mathbf{u}}$ in $L^2((0,T)\times\Omega)^d$. In order to prove Theorem 1, it only remains to study the convergence of the gradient part of the velocity $Q\mathbf{u}^\varepsilon$.

3.1. Some properties of acoustic modes

First, we want to solve the spectral problem associated with the viscous wave operator A_ε in terms of eigenvalues and eigenvectors of the inviscid wave operator A_0. In the sequel, we will denote the density fluctuation $\Psi^\varepsilon := (\rho^\varepsilon - 1)/\varepsilon$, and $\phi^\varepsilon = (\Psi^\varepsilon, \mathbf{m}^\varepsilon)^t$. The wave operators A_0 and A_ε are defined on $\mathcal{D}'(\Omega)\times\mathcal{D}'(\Omega)^d$ by

$$A_0\begin{pmatrix}\Psi\\ \mathbf{m}\end{pmatrix} = \begin{pmatrix}\operatorname{div}\mathbf{m}\\ \nabla\Psi\end{pmatrix}, \tag{12}$$

and

$$A_\varepsilon\begin{pmatrix}\Psi\\ \mathbf{m}\end{pmatrix} = A_0\begin{pmatrix}\Psi\\ \mathbf{m}\end{pmatrix} + \varepsilon\begin{pmatrix}0\\ \mu\Delta\mathbf{m} + (\lambda+\mu)\nabla\operatorname{div}\mathbf{m}\end{pmatrix}. \tag{13}$$

The eigenvalues and eigenvectors of A_0 read as follows

$$\phi_{k,0}^\pm = \left(\Psi_{k,0}, \mathbf{m}_{k,0}^\pm = \pm\frac{\nabla\Psi_{k,0}}{\mathrm{i}\lambda_{k,0}}\right)^t \in \mathbb{C}^{1+d},$$

$$A_0\phi_{k,0}^\pm = \pm\mathrm{i}\,\lambda_{k,0}\phi_{k,0}^\pm \quad \text{in } \Omega, \qquad \mathbf{m}_{k,0}^\pm\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{14}$$

We want to prove:

PROPOSITION 2. – *Let Ω be a C^2 bounded domain of $\mathbb{R}^d$ and let $k \geqslant 1$, $N \geqslant 0$. Then, there exists approximate eigenvalues $\mathrm{i}\lambda^{\pm}_{k,\varepsilon,N}$ and eigenvectors $\phi^{\pm}_{k,\varepsilon,N} = (\Psi^{\pm}_{k,\varepsilon,N}, \mathbf{m}^{\pm}_{k,\varepsilon,N})^t$ of A_ε such that*

$$A_\varepsilon \phi^{\pm}_{k,\varepsilon,N} = \mathrm{i}\lambda^{\pm}_{k,\varepsilon,N}\phi^{\pm}_{k,\varepsilon,N} + R^{\pm}_{k,\varepsilon,N}, \tag{15}$$

with

$$\mathrm{i}\lambda^{\pm}_{k,\varepsilon,N} = \pm \mathrm{i}\lambda_{k,0} + \mathrm{i}\lambda^{\pm}_{k,1}\sqrt{\varepsilon} + \mathrm{O}(\varepsilon), \quad \textit{where } \mathcal{R}e\big(\mathrm{i}\lambda^{\pm}_{k,1}\big) \leqslant 0, \tag{16}$$

and for all $p \in [1,\infty]$, we have:

$$\big|R^{\pm}_{k,\varepsilon,N}\big|_{L^p(\Omega)} \leqslant C_p\big(\sqrt{\varepsilon}\big)^{N+1/p} \quad \textit{and} \quad \big|\phi^{\pm}_{k,\varepsilon,N} - \phi^{\pm}_{k,0}\big|_{L^p(\Omega)} \leqslant C_p\big(\sqrt{\varepsilon}\big)^{1/p}. \tag{17}$$

Proof. – The main idea is to build approximate modes of A_ε in terms of $\phi^{\pm}_{k,0}$. More precisely, we make for $\phi^{\pm}_{k,\varepsilon,N}$ and $\lambda^{\pm}_{k,\varepsilon,N}$ the following ansatz

$$\phi^{\pm}_{k,\varepsilon,N}(\mathbf{x}) = \sum_{i=0}^{N}\left(\sqrt{\varepsilon}^{\,i}\phi^{\pm,int}_{k,i}(\mathbf{x}) + \sqrt{\varepsilon}^{\,i}\phi^{\pm,b}_{k,i}\left(\mathbf{x}, \frac{d(\mathbf{x})}{\sqrt{\varepsilon}}\right)\right), \tag{18}$$

$$\lambda^{\pm}_{k,\varepsilon,N} = \sum_{i=0}^{N}\sqrt{\varepsilon}^{\,i}\lambda^{\pm}_{k,i}, \tag{19}$$

where $\phi^{\pm,int}_{k,i} = (\Psi^{\pm,int}_{k,i}, \mathbf{m}^{\pm,int}_{k,i})$ and $\phi^{\pm,b}_{k,i} = (\Psi^{\pm,b}_{k,i}, \mathbf{m}^{\pm,b}_{k,i})$ are smooth functions such that $\mathbf{m}^{\pm,int}_{k,j} + \mathbf{m}^{\pm,b}_{k,j} = 0$ on $\partial\Omega$, $\phi^{\pm,b}_{k,i}$ being rapidly decreasing to 0 in the ζ variable defined by $\zeta = d(\mathbf{x})/\sqrt{\varepsilon}$. Here, d denotes a regularized distance function to the boundary $\partial\Omega$, i.e., any function $d \in C^2(\Omega)$ such that $d > 0$ in Ω, $d = 0$ and $\nabla d = -\mathbf{n}$ on $\partial\Omega$. We first find $\phi^{\pm,int}_{k,0} = \phi^{\pm}_{k,0}$ and $\lambda^{\pm}_{k,0} = \pm\lambda_{k,0}$. Since $\mathbf{m}^{\pm,int}_{k,0} \neq 0$ on the boundary $\partial\Omega$, we introduce a boundary layer corrector $\mathbf{m}^{\pm,b}_{k,0}$ defined by identifying terms of order $\sqrt{\varepsilon}^{\,-1}$ and $\sqrt{\varepsilon}^{\,0}$ in (15). Order $\sqrt{\varepsilon}^{\,-1}$ in the equation on $\Psi^{\pm}_{k,\varepsilon,N}$ gives

$$\partial_\zeta \mathbf{m}^{\pm,b}_{k,0} \cdot \nabla d = 0 \quad \text{hence} \quad \mathbf{m}^{\pm,b}_{k,0} \cdot \nabla d = 0.$$

Order $\sqrt{\varepsilon}^{\,-1}$ in the equation on $\mathbf{m}^{\pm}_{k,\varepsilon,N}$ yields

$$\partial_\zeta \Psi^{\pm,b}_{k,0}\nabla d = 0 \quad \text{hence} \quad \Psi^{\pm,b}_{k,0} \equiv 0.$$

Order $\sqrt{\varepsilon}^{\,0}$ in the equation on $\mathbf{m}^{\pm}_{k,\varepsilon,N}$ gives

$$\partial_\zeta \Psi^{\pm,b}_{k,1}\nabla d + \mu\partial_\zeta^2\mathbf{m}^{\pm,b}_{k,0}|\nabla d|^2 + (\lambda+\mu)\big(\partial_\zeta^2\mathbf{m}^{\pm,b}_{k,0}\cdot\nabla d\big)\nabla d = \mathrm{i}\lambda^{\pm}_{k,0}\mathbf{m}^{\pm,b}_{k,0}, \tag{20}$$

using the fact that $\mathbf{m}^{\pm,b}_{k,0} \cdot \nabla d = 0$ and taking the scalar product of (20) with ∇d, we get

$$\partial_\zeta \Psi^{\pm,b}_{k,1} = 0 \quad \text{hence} \quad \Psi^{\pm,b}_{k,1} \equiv 0. \tag{21}$$

As a result, (20) gives

$$\mu|\nabla d|^2\partial_\zeta^2\mathbf{m}_{k,0}^{\pm,b}=\pm\mathrm{i}\lambda_{k,0}\mathbf{m}_{k,0}^{\pm,b}. \tag{22}$$

Keeping in mind that $\mathbf{m}_{k,0}^{\pm,b}+\mathbf{m}_{k,0}^{\pm,int}=0$ on $\zeta=0$, we obtain

$$\mathbf{m}_{k,0}^{\pm,b}(\mathbf{x},\zeta)\times\nabla d=-\big(\mathbf{m}_{k,0}^{\pm,int}(\mathbf{x})\times\nabla d\big)\exp\left(-\zeta\,\frac{(1\pm\mathrm{i})}{|\nabla d|}\sqrt{\frac{\lambda_{k,0}}{2\mu}}\right). \tag{23}$$

Using the equation on $\Psi_{k,\varepsilon,N}^{\pm}$ at order $\sqrt{\varepsilon}^{\,0}$, we obtain the following equation

$$\partial_\zeta\mathbf{m}_{k,1}^{\pm,b}\cdot\nabla d=-\operatorname{div}\mathbf{m}_{k,0}^{\pm,b}.$$

An integration in ζ then yields the expression of $\mathbf{m}_{k,1}^{\pm,b}\cdot\nabla d$, which, denoting Δ_g the Laplace–Beltrami operator on $\partial\Omega$, gives in particular

$$\mathbf{m}_{k,1}^{\pm,b}\cdot\mathbf{n}=-(1\pm\mathrm{i})(\Delta_g\Psi_{k,0})\sqrt{\frac{\mu}{2\lambda_{k,0}^3}}\quad\text{on }\partial\Omega. \tag{24}$$

We next build $\Psi_{k,1}^{\pm,int}$, $\mathbf{m}_{k,1}^{\pm,int}$, and $\lambda_{k,1}^{\pm}$ such that:

$$\operatorname{div}\mathbf{m}_{k,1}^{\pm,int}=\mathrm{i}\lambda_{k,0}^{\pm}\Psi_{k,1}^{\pm,int}+\mathrm{i}\lambda_{k,1}^{\pm}\Psi_{k,0},\quad\text{and}\quad\nabla\Psi_{k,1}^{\pm,int}=\mathrm{i}\lambda_{k,0}^{\pm}\mathbf{m}_{k,1}^{\pm,int}+\mathrm{i}\lambda_{k,1}^{\pm}\mathbf{m}_{k,0}^{\pm,int},$$

with $\mathbf{m}_{k,1}^{\pm,int}\cdot\mathbf{n}=-\mathbf{m}_{k,1}^{\pm,b}\cdot\mathbf{n}$ on the boundary, which can be rewritten in terms of $\Psi_{k,1}^{\pm,int}$ as follows

$$-\Delta\Psi_{k,1}^{\pm,int}=\lambda_{k,0}^2\Psi_{k,1}^{\pm,int}+2\lambda_{k,0}^{\pm}\lambda_{k,1}^{\pm}\Psi_{k,0}\quad\text{in }\Omega, \tag{25}$$

and

$$\partial_{\mathbf{n}}\Psi_{k,1}^{\pm,int}=-\mathrm{i}\lambda_{k,0}^{\pm}\mathbf{m}_{k,1}^{\pm,b}\cdot\mathbf{n}\quad\text{on }\partial\Omega. \tag{26}$$

Taking the scalar product of (25) with $\Psi_{k,0}$ yields

$$\mathrm{i}\lambda_{k,1}^{\pm}=-\frac{(1\pm\mathrm{i})}{2}\sqrt{\frac{\mu}{2\lambda_{k,0}^3}}\int_{\partial\Omega}|\nabla\Psi_{k,0}|^2\,\mathrm{d}\sigma,\quad\text{which satisfies}\quad\mathcal{R}e\big(\mathrm{i}\lambda_{k,1}^{\pm}\big)\leqslant 0. \tag{27}$$

Taking similarly the scalar product of (25) with $\Psi_{l,0}\neq\Psi_{k,0}$, we obtain

$$-\int_{\partial\Omega}\Psi_{l,0}\,\partial_{\mathbf{n}}\Psi_{k,1}^{\pm,int}\,\mathrm{d}\sigma=\big(\lambda_{k,0}^2-\lambda_{l,0}^2\big)\int_{\Omega}\Psi_{k,1}^{\pm,int}\Psi_{l,0}\,\mathrm{d}\mathbf{x}, \tag{28}$$

which determines $\langle\Psi_{k,1}^{\pm,int}\mid\Psi_{l,0}\rangle$ if $\lambda_{k,0}\neq\lambda_{l,0}$. If $\lambda_{k,0}=\lambda_{l,0}$ condition (10) implies that the left hand side of (28) vanishes, which is the case indeed, and $\langle\Psi_{k,1}^{\pm,int}\mid\Psi_{l,0}\rangle$ can be chosen arbitrarily at this stage. It is in fact determined by the next order of the ansatz. Similarly, the expressions of $\mathbf{m}_{k,1}^{\pm,b}\times\nabla d$ and $\Psi_{k,2}^{\pm,b}$ stem from

$$\partial_\zeta \Psi^{\pm,b}_{k,2} \nabla d + \mu \partial^2_\zeta \mathbf{m}^{\pm,b}_{k,1} |\nabla d|^2 + (\lambda+\mu)\big(\partial^2_\zeta \mathbf{m}^{\pm,b}_{k,1} \cdot \nabla d\big) \nabla d - \mathrm{i}\lambda^{\pm}_{k,0} \mathbf{m}^{\pm,b}_{k,1}$$
$$= \mathrm{i}\lambda^{\pm}_{k,1} \mathbf{m}^{\pm,b}_{k,0} - 2\mu \nabla^j \partial_\zeta \mathbf{m}^{\pm,b}_{k,0} \cdot \nabla^j d, \tag{29}$$

which allows to keep building approximate solutions at any order. □

Let us observe that after a time rescaling of order ε, the first order term $\mathrm{i}\lambda^{\pm}_{k,1}$ clearly yields an instantaneous damping of acoustic waves, as soon as $\mathcal{R}e(\mathrm{i}\lambda^{\pm}_{k,1}) < 0$. For this reason, we define $I \subset \mathbb{N}$ to be the set of eigenmodes $\Psi_{k,0}$ of $-\Delta_N$ such that $\mathcal{R}e(\mathrm{i}\lambda^{\pm}_{k,1}) < 0$ and $J = \mathbb{N} \setminus I$. Notice that when $k \in J$, we have $\lambda^{\pm}_{k,1} = 0$. For those indices, $\mathbf{m}^{\pm}_{k,0}$ identically vanishes on $\partial\Omega$ and therefore satisfies not only $\mathbf{m}^{\pm}_{k,0} \cdot \mathbf{n} = 0$ but also $\mathbf{m}^{\pm}_{k,0} = 0$ on $\partial\Omega$, hence no significant boundary layer is created, and there is no enhanced dissipation of energy in these layers.

3.2. Conclusion

Let us finally study the convergence properties of $Q\mathbf{u}^\varepsilon$. First of all, we write $Q\mathbf{u}^\varepsilon = \nabla \Lambda^\varepsilon$, where $\Lambda^\varepsilon \in L^2(0,T;H^2(\Omega))$ is defined by:

$$-\Delta \Lambda^\varepsilon = -\operatorname{div} \mathbf{u}^\varepsilon \quad \text{in } \Omega, \qquad \frac{\partial \Lambda^\varepsilon}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \quad \text{and} \quad \int_\Omega \Lambda^\varepsilon \,\mathrm{d}\mathbf{x} = 0.$$

Writing Λ^ε on the orthonormal basis $(\Psi_{k,0})_{k\geqslant 0}$ of $L^2(\Omega)$ functions with zero mean value on Ω, we obtain an expansion of $Q\mathbf{u}^\varepsilon$ on the orthonormal family $(\nabla\Psi_{k,0}/\lambda_{k,0})_{k\geqslant 0}$

$$Q\mathbf{u}^\varepsilon = \sum_{k\in\mathbb{N}} \left\langle Q\mathbf{u}^\varepsilon \;\middle|\; \frac{\nabla\Psi_{k,0}}{\lambda_{k,0}} \right\rangle \frac{\nabla\Psi_{k,0}}{\lambda_{k,0}}.$$

We now split $Q\mathbf{u}^\varepsilon$ into $Q_1\mathbf{u}^\varepsilon$ and $Q_2\mathbf{u}^\varepsilon$, defined by

$$Q_1\mathbf{u}^\varepsilon = \sum_{k\in I} \left\langle Q\mathbf{u}^\varepsilon \;\middle|\; \frac{\nabla\Psi_{k,0}}{\lambda_{k,0}} \right\rangle \frac{\nabla\Psi_{k,0}}{\lambda_{k,0}} \quad \text{and} \quad Q_2\mathbf{u}^\varepsilon = \sum_{k\in J} \left\langle Q\mathbf{u}^\varepsilon \;\middle|\; \frac{\nabla\Psi_{k,0}}{\lambda_{k,0}} \right\rangle \frac{\nabla\Psi_{k,0}}{\lambda_{k,0}},$$

which respectively correspond to damped modes and nondamped modes. We will prove on the one hand that $Q_1\mathbf{u}^\varepsilon$ converges strongly to 0 in $L^2((0,T)\times\Omega)^d$, and on the other hand that $\operatorname{curl}\operatorname{div}(Q_2\mathbf{m}^\varepsilon \otimes Q_2\mathbf{u}^\varepsilon)$ converges to 0 in the sense of distributions, if $J \neq \emptyset$.

Let us observe that in view of the $L^2(0,T;H^1_0(\Omega))^d$ bound on $\mathbf{u}^\varepsilon$, the problem reduces to a finite number of modes. Indeed, we have

$$\sum_{k>N} \int_0^T \left| \left\langle Q_i\mathbf{u}^\varepsilon \;\middle|\; \frac{\nabla\Psi_{k,0}}{\lambda_{k,0}} \right\rangle \right|^2 \mathrm{d}t \leqslant \frac{C}{\lambda^2_{N+1}} \left|\nabla \mathbf{u}^\varepsilon\right|^2_{L^2((0,T)\times\Omega)}, \quad i = 1 \text{ or } 2.$$

Hence recalling that $\lambda_N \to +\infty$ as $N \to +\infty$, we only have to prove that $\langle Q_1\mathbf{u}^\varepsilon \mid \mathbf{m}^{\pm}_{k,0}\rangle$ converges strongly to 0 in $L^2(0,T)$ for any fixed k, and study the interaction of a finite number of modes in $\operatorname{div}(Q_2\mathbf{u}^\varepsilon \otimes Q_2\mathbf{u}^\varepsilon)$ using the group method like in [6,15].

Let us next make a series of remarks. First, we have the identity $Q\mathbf{u}^\varepsilon = Q\mathbf{m}^\varepsilon - \varepsilon Q(\Psi^\varepsilon \mathbf{u}^\varepsilon)$, and

$$\varepsilon\left|\left\langle Q\left(\Psi^\varepsilon \mathbf{u}^\varepsilon\right) \mid \nabla\Psi_{k,0}\right\rangle\right| = \varepsilon\left|\int_\Omega \Psi^\varepsilon \mathbf{u}^\varepsilon \cdot \nabla\Psi_{k,0}\,\mathrm{d}\mathbf{x}\right| \leqslant \varepsilon\left|\Psi^\varepsilon\right|_{L^\gamma(\Omega)}\left|\mathbf{u}^\varepsilon\right|_{L^{\frac{\gamma}{\gamma-1}}(\Omega)}\left|\nabla\Psi_{k,0}\right|_{L^\infty(\Omega)},$$

which goes to 0 in $L^2(0,T)$ since $1-1/\gamma < 1/2 - 1/d$ and $\gamma > d/2$. Hence, we are led to study $\langle Q\mathbf{m}^\varepsilon \mid \mathbf{m}^\pm_{k,0}\rangle$.

Defining $\beta^\pm_{k,\varepsilon}(t) = \langle \phi^\varepsilon(t) \mid \phi^\pm_{k,0}\rangle$, we observe that: $2\langle Q\mathbf{m}^\varepsilon \mid \mathbf{m}^\pm_{k,0}\rangle = \beta^\pm_{k,\varepsilon} - \beta^\mp_{k,\varepsilon}$, so that it suffices to consider the convergence properties of $\beta^\pm_{k,\varepsilon}$ in $L^2(0,T)$.

As a last remark, we deduce from Proposition 2 applied with $N=2$ that:

$$\left|\left\langle \phi^\varepsilon(t) \mid \phi^\pm_{k,0} - \phi^\pm_{k,\varepsilon,2}\right\rangle\right| \leqslant C\sqrt{\varepsilon}^{\,\alpha}\left(\left|\Psi^\varepsilon\right|_{L^\infty(0,T;L^\kappa(\Omega))} + \left|\mathbf{m}^\varepsilon\right|_{L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\Omega))}\right),$$

where $\alpha = \inf(1-1/\kappa, 1/2 - 1/2\gamma)$, hence we only have to prove that $b^\pm_{k,\varepsilon}(t) = \langle \phi^\varepsilon(t) \mid \phi^\pm_{k,\varepsilon,2}\rangle$ converges strongly to 0 in $L^2(0,T)$ when $k \in I$, and study its oscillations when $k \in J$.

Notice that $\phi^\varepsilon = (\Psi^\varepsilon, \mathbf{m}^\varepsilon)^t$ solves

$$\partial_t\phi^\varepsilon - \frac{A^*_\varepsilon\phi^\varepsilon}{\varepsilon} = \begin{pmatrix} 0 \\ \mathbf{g}^\varepsilon \end{pmatrix}, \tag{30}$$

where A^*_ε denotes the adjoint of A_ε with respect to $\langle . \mid . \rangle$, and

$$\mathbf{g}^\varepsilon = -\varepsilon\mu\Delta\left(\mathbf{u}^\varepsilon\Psi^\varepsilon\right) - \varepsilon(\lambda+\mu)\nabla\operatorname{div}\left(\mathbf{u}^\varepsilon\Psi^\varepsilon\right) - \operatorname{div}\left(\mathbf{m}^\varepsilon \otimes \mathbf{u}^\varepsilon\right) - (\gamma-1)\nabla\pi^\varepsilon. \tag{31}$$

Taking the scalar product of (30) with $\phi^\pm_{k,\varepsilon,2}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} b^\pm_{k,\varepsilon}(t) - \frac{\overline{\mathrm{i}\lambda^\pm_{k,\varepsilon,2}}}{\varepsilon} b^\pm_{k,\varepsilon}(t) = c^\pm_{k,\varepsilon}(t), \tag{32}$$

where $c^\pm_{k,\varepsilon}(t) = \langle \mathbf{g}^\varepsilon \mid \mathbf{m}^\pm_{k,\varepsilon,2}\rangle + \varepsilon^{-1}\langle \phi^\varepsilon \mid R^\pm_{k,\varepsilon,2}\rangle$.

The case $k \in I$:
From (32), we deduce that

$$b^\pm_{k,\varepsilon}(t) = b^\pm_{k,\varepsilon}(0)\mathrm{e}^{\overline{\mathrm{i}\lambda^\pm_{k,\varepsilon,2}}t/\varepsilon} + \int_0^t c^\pm_{k,\varepsilon}(s)\,\mathrm{e}^{\overline{\mathrm{i}\lambda^\pm_{k,\varepsilon,2}}(t-s)/\varepsilon}\,\mathrm{d}s. \tag{33}$$

The first term in (33) is estimated as follows

$$\left|\mathrm{e}^{\overline{\mathrm{i}\lambda^\pm_{k,\varepsilon,2}}t/\varepsilon} b^\pm_{k,\varepsilon}(0)\right|_{L^2(0,T)} \leqslant C\left|\mathrm{e}^{\mathcal{R}e(\overline{\mathrm{i}\lambda^\pm_{k,1}})t/\sqrt{\varepsilon}} b^\pm_{k,\varepsilon}(0)\right|_{L^2(0,T)} \leqslant C\varepsilon^{1/4}.$$

In order to estimate the remaining term in (33), we observe that for any $a \in L^q(0,T)$ and $1 \leqslant p,q \leqslant \infty$ such that $p^{-1}+q^{-1}=1$, we have

$$\left|\int_0^t \mathrm{e}^{\overline{\mathrm{i}\lambda^\pm_{k,\varepsilon,2}}(t-s)/\varepsilon} a(s)\,\mathrm{d}s\right| \leqslant \int_0^t \mathrm{e}^{\mathcal{R}e(\overline{\mathrm{i}\lambda^\pm_{k,1}})(t-s)/\sqrt{\varepsilon}}\left|a(s)\right|\mathrm{d}s \leqslant C|a|_{L^q(0,T)}\sqrt{\varepsilon}^{\,1/p}. \tag{34}$$

We now write $|c_{k,\varepsilon}^{\pm}| \leqslant c_1 + (\gamma-1)c_2 + c_3 + c_4$, where

$$c_1(t) = \left|\int_\Omega \big(\mathbf{m}^\varepsilon \otimes \mathbf{u}^\varepsilon\big)(t)\cdot \nabla \mathbf{m}_{k,\varepsilon,2}^{\pm}\,\mathrm{d}\mathbf{x}\right|, \qquad c_2(t) = \left|\int_\Omega \pi^\varepsilon(t)\,\mathrm{div}\,\mathbf{m}_{k,\varepsilon,2}^{\pm}\,\mathrm{d}\mathbf{x}\right|,$$

$$c_3(t) = \varepsilon\left|\int_\Omega \mathbf{m}_{k,\varepsilon,2}^{\pm}\cdot\big(\mu\Delta + (\lambda+\mu)\nabla\,\mathrm{div}\big)\big(\mathbf{u}^\varepsilon\Psi^\varepsilon\big)(t)\right| \quad \text{and}$$

$$c_4(t) = \varepsilon^{-1}\big|\langle \phi^\varepsilon \mid R_{k,\varepsilon,2}^{\pm}\rangle\big|.$$

Observing that $\mathbf{m}^\varepsilon = \varepsilon\Psi^\varepsilon\mathbf{u}^\varepsilon + \mathbf{u}^\varepsilon$, we have:

$$\begin{aligned} c_1(t) \leqslant{}& \big|\mathbf{m}_{k,\varepsilon,2}^{\pm}\big|_{L^\infty(\Omega)}\big|\mathbf{u}_1^\varepsilon + \mathbf{u}_2^\varepsilon\big|_{L^2(\Omega)}\big|\nabla\mathbf{u}^\varepsilon\big|_{L^2(\Omega)} \\ &+ \varepsilon\big|\Psi^\varepsilon\big|_{L^\infty(0,T;L^\kappa(\Omega))}\big|\mathbf{u}^\varepsilon\big|_{L^{\kappa/(\kappa-1)}(\Omega)}\big|\nabla\mathbf{m}_{k,\varepsilon,2}^{\pm}\big|_{L^\infty(\Omega)} \\ \leqslant{}& C\big|\mathbf{u}_1^\varepsilon\big|_{L^\infty(0,T;L^2(\Omega))}\big|\nabla\mathbf{u}^\varepsilon\big|_{L^2(\Omega)} + C\varepsilon^{1/2}\big|\nabla\mathbf{u}^\varepsilon\big|_{L^2(\Omega)}^2 + C\varepsilon^{1/2}\big|\nabla\mathbf{u}^\varepsilon\big|_{L^2(\Omega)}, \end{aligned}$$

using the decomposition $\mathbf{u}^\varepsilon = \mathbf{u}_1^\varepsilon + \mathbf{u}_2^\varepsilon$ (see Section 2). The second term c_2 is estimated recalling (15)

$$c_2(t) \leqslant C\big|\pi^\varepsilon\big|_{L^\infty(0,T;L^1(\Omega))}\big(\big|\Psi_{k,\varepsilon,2}^{\pm}\big|_{L^\infty(\Omega)} + \big|R_{k,\varepsilon,2}^{\pm}\big|_{L^\infty(\Omega)}\big) \leqslant C.$$

Using the fact that $\mathbf{m}_{k,\varepsilon,2}^{\pm}$ and $\Psi^\varepsilon\mathbf{u}^\varepsilon$ satisfy homogeneous Dirichlet boundary conditions, we deduce that

$$\begin{aligned} c_3(t) &\leqslant \varepsilon\left|\int_\Omega \big(\mu\Delta + (\lambda+\mu)\nabla\,\mathrm{div}\big)\mathbf{m}_{k,\varepsilon,2}^{\pm}\cdot\mathbf{u}^\varepsilon\Psi^\varepsilon\,\mathrm{d}\mathbf{x}\right| \\ &\leqslant \varepsilon\big|D^2\mathbf{m}_{k,\varepsilon,2}^{\pm}\big|_{L^\infty(\Omega)}\big|\mathbf{u}^\varepsilon\big|_{L^{\kappa/(\kappa-1)}(\Omega)}\big|\Psi^\varepsilon\big|_{L^\infty(0,T;L^\kappa(\Omega))} \leqslant C\big|\nabla\mathbf{u}^\varepsilon\big|_{L^2(\Omega)}. \end{aligned}$$

Finally, we estimate c_4

$$c_4(t) \leqslant \frac{1}{\varepsilon}\big|R_{k,\varepsilon,2}^{\pm}\big|_{L^{\kappa/(\kappa-1)}(\Omega)}\big|\phi^\varepsilon\big|_{L^\infty(0,T;L^\kappa(\Omega))} \leqslant C\varepsilon^{1/2-1/2\kappa}.$$

It is now easy to conclude making extensive use of (34) that $b_{k,\varepsilon}^{\pm}$ converges strongly to 0 in $L^2(0,T)$.

The case $k \in J$:
From (32) and the fact that $\lambda_{k,1}^{\pm} = 0$, we deduce that $\mathrm{e}^{\mathrm{i}\lambda_{k,0}^{\pm}t/\varepsilon}b_{k,\varepsilon}^{\pm}$ is bounded in $L^2(0,T)$ and that its derivative in time is bounded in $\sqrt{\varepsilon}L^1(0,T) + L^p(0,T)$ for some $p > 1$. It follows that up to the extraction of a subsequence, it converges strongly in $L^2(0,T)$ to some $\bar{b}_{k,osc}^{\pm}$. As a result, the problem reduces to study interaction of resonant time oscillations, so that we conclude as in [6,15] by observing that as long as $\lambda_{k,0} = \lambda_{l,0}$, we have

$$\mathrm{div}(\nabla\Psi_{k,0}\otimes\nabla\Psi_{l,0} + \nabla\Psi_{l,0}\otimes\nabla\Psi_{k,0}) = -\lambda_{k,0}^2\nabla(\Psi_{k,0}\Psi_{l,0}) + \nabla(\nabla\Psi_{k,0}\cdot\nabla\Psi_{l,0}),$$

which is a gradient, and thus disappears in the pressure term. Using the methods of [6,15], it is possible to write down an equation for $\mathrm{e}^{\mathrm{i}\lambda_{k,0}^{\pm}t/\varepsilon}b_{k,\varepsilon}^{\pm}$, which describes the evolution of the acoustic waves, their mutual interactions, and their interactions with the limit flow. We will not enter into the details of the algebra and refer to ([6,15,19]) for more details.

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